

# A Short Note on Two-loop Box Functions

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## Abstract

It is shown that the two-loop four-point functions are similar in structure to the three-point two-loop functions for all mass cases and topologies. The result is derived by using a rotation to a  $(+, -, -, +)$  signature without spoiling analyticity properties.

In this short note we will discuss some similarities between two-loop three- and four-point functions. It was first noticed in [1] that there is a striking similarity between the special functions appearing in the three- and four-point cases. But this result was derived only for the very restricted mass case of vanishing internal masses. In this note we will show that the two-loop box functions allow for integral representations similar to the ones obtained for the three-point case in all mass cases and for all possible topologies. Following the lines of [2] we hope to be able to come up with explicit numerical results for arbitrary two-loop box functions in the future [3].

The paper is organized as follows. We first show how to transform the integrals via an unconventional Wick rotation to a  $(+, -, -, +)$  signature. We explicitly show that this will not spoil any analyticity properties of the Green functions. In the next step, we show that the three-point and four-point function (and also the two-point function) can be treated on the same footing. This is based on the fact that both cases allow for similar two- and three-particle cuts. The results turn out to be three-fold integral representations similar to the ones obtained in [4] for the three-point functions. These can then be transformed into convenient integral representations along the lines of [2].

All possible two-loop functions can be derived from the following topology Fig.(1) by assigning two, three, or four external particles in all possible ways to it.

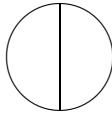


Fig.1: The generic two-loop topology.

Consider for example the four-point functions. We have four different topologies:

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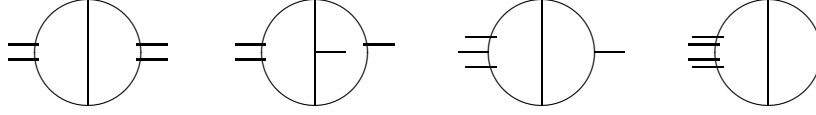


Fig.2a,...,d: The two-loop four-point topologies.

They can all be written in the form (our only objective being scalar integrals)

$$\int d^4l \int d^4k \frac{1}{N_l} \frac{1}{N_{l,k}} \frac{1}{N_k} \quad (1)$$

where

- for Fig.(2a) we have:

$$\begin{aligned} N_l &= (l^2 - m_1^2 + i\eta)((l + p_1)^2 - m_2^2 + i\eta)((l + p_1 + p_2)^2 - m_3^2 + i\eta) \\ N_{l,k} &= (l + k)^2 - m_4^2 + i\eta \\ N_k &= (k^2 - m_5^2 + i\eta)((k + p_3)^2 - m_6^2 + i\eta)((k - p_1 - p_2)^2 - m_7^2 + i\eta), \end{aligned}$$

- for Fig.(2b) we have:

$$\begin{aligned} N_l &= (l^2 - m_1^2 + i\eta)((l + p_1)^2 - m_2^2 + i\eta)((l + p_1 + p_2)^2 - m_3^2 + i\eta) \\ N_{l,k} &= ((l + k)^2 - m_4^2 + i\eta)((l + k + p_3)^2 - m_5^2 + i\eta) \\ N_k &= ((k + p_3)^2 - m_6^2 + i\eta)((k - p_1 - p_2)^2 - m_7^2 + i\eta), \end{aligned}$$

- for Fig.(2c) we have

$$\begin{aligned} N_l &= (l^2 - m_1^2 + i\eta)((l + p_1)^2 - m_2^2 + i\eta)((l + p_1 + p_2)^2 - m_3^2 + i\eta) \\ &\quad ((l + p_1 + p_2 + p_3)^2 - m_4^2 + i\eta) \\ N_{l,k} &= (l + k)^2 - m_5^2 + i\eta \\ N_k &= (k^2 - m_6^2 + i\eta)((k - p_1 - p_2 - p_3)^2 - m_7^2 + i\eta), \end{aligned}$$

- while for Fig.(2d) we have

$$\begin{aligned} N_l &= (l^2 - m_1^2 + i\eta)((l + p_1)^2 - m_2^2 + i\eta)((l + p_1 + p_2)^2 - m_3^2 + i\eta) \\ &\quad ((l + p_1 + p_2 + p_3)^2 - m_4^2 + i\eta)(l^2 - m_5^2 + i\eta) \\ N_{l,k} &= (l + k)^2 - m_6^2 + i\eta \\ N_k &= (k^2 - m_7^2 + i\eta) + i\eta. \end{aligned}$$

Here and in the following it is always understood that the integration is split into an integration over the span of the exterior momenta (the parallel space) and an integration over its orthogonal complement (the orthogonal space) [5, 4]. The propagators are quadratic forms in four resp. eight variables each. Note that the fourth variable ( $l_3$  resp.  $k_3$  which we can assume to coincide with the orthogonal space variables) does not mix with components of exterior momenta. As functions of  $l_3, k_3$  our denominators behave as

$$\begin{aligned} N_l &= N_l(l_3^2) \\ N_{l,k} &= N_{l,k}((l_3 + k_3)^2) \\ N_k &= N_k(k_3^2). \end{aligned}$$

We further stress that also the two-loop two- and three-point functions fit into the template of Eq.(1). If we consider the number of propagators and the number of variables that

do not mix with exterior momenta in the propagators, it is only in these numbers where the two-, three- and four-point functions differ from our viewpoint.

Now let us do the following transformations:

$$\begin{aligned}
& \int_{-\infty}^{+\infty} dl_3 \int_{-\infty}^{+\infty} dk_3 \frac{1}{N_l(l_3^2)N_{l,k}((l_3 + k_3)^2)N_k(k_3^2)} \\
&= 2 \int_0^{+\infty} dl_3 \int_0^{+\infty} dk_3 \left( \frac{1}{N_l(l_3^2)N_{l,k}((l_3 + k_3)^2)N_k(k_3^2)} \right. \\
&\quad \left. + \frac{1}{N_l(l_3^2)N_{l,k}((l_3 - k_3)^2)N_k(k_3^2)} \right) \\
&= \frac{1}{2} \int_0^{+\infty} \frac{ds}{\sqrt{s}} \int_0^{+\infty} \frac{dt}{\sqrt{t}} \left( \frac{1}{N_l(s)N_{l,k}((\sqrt{s} + \sqrt{t})^2)N_k(t)} \right. \\
&\quad \left. + \frac{1}{N_l(s)N_{l,k}((\sqrt{s} - \sqrt{t})^2)N_k(t)} \right) \\
&= \int_0^1 du \int_0^{+\infty} v dv \left( \frac{1}{N_l(uv^2)N_{l,k}(v^2(\sqrt{u} + \sqrt{1-u})^2)N_k((1-u)v^2)} \right. \\
&\quad \left. + \frac{1}{N_l(uv^2)N_{l,k}(v^2(\sqrt{u} - \sqrt{1-u})^2)N_k((1-u)v^2)} \right).
\end{aligned}$$

Using that  $(u, (1-u), (\sqrt{u} \pm \sqrt{1-u})^2)$  all are  $\geq 0$  between zero and one we see that, for all propagators, the poles in the  $v$  variable are located in the first or third quadrant (in the complex  $v$ -plane). So we rotate  $\pi/2$  clockwise in the  $v$ -plane and invert all the above transformations to obtain

$$\begin{aligned}
& \int_{-\infty}^{+\infty} dl_3 \int_{-\infty}^{+\infty} dk_3 \frac{1}{N_l(l_3^2)N_{l,k}((l_3 + k_3)^2)N_k(k_3^2)} \\
&= \int_{-\infty}^{+\infty} dl_3 \int_{-\infty}^{+\infty} dk_3 \frac{1}{N_l(-l_3^2)N_{l,k}(-(l_3 + k_3)^2)N_k(-k_3^2)}.
\end{aligned}$$

We see that we effectively have transformed our Green functions from a metric with signature  $(+, -, -, -)$  to a signature  $(+, -, -, +)$ . As this is quite an unusual transformation we have given the explicit derivation above to show that this is possible without spoiling analyticity properties. Note further that this does not restrict the domain of validity of our results. This is in contrast with the standard Wick rotation [6]:

There one first considers an euclidean domain  $q_i^2 < 0$  for all exterior momenta  $q_i$ . That guarantees that all these euclidean vectors are orthogonal to some timelike vector  $n$  say. Lorentz invariance of the Green functions allows to choose  $n = (1, 0, 0, 0)$ . In this frame the 0-component of all euclidean vectors  $q_i$  vanishes, so that all propagators are quadratic forms where the 0-variables ( $l_0$  resp.  $k_0$ ) do not mix with exterior momenta. This then guarantees that the causal behaviour of all propagators is such that the poles of all propagators are located either in the second or fourth quadrant of the complex  $l_0$ - resp.  $k_0$ -plane. Consequently, this allows for a Wick rotation in these variables, which would otherwise be spoilt by the 0-components of the exterior momenta. Indeed, these components would shift the poles into the other quadrants. So one ends up with the result that for exterior euclidean momenta one can Wick rotate to a metric of definite signature. One recovers then the Green function for arbitrary timelike exterior momenta by an appropriate analytic continuation. With this step one touches some fundamental results of field theory, e.g. the ability to recover the Wightman functions from the Schwinger functions [7].

In our case the situation is much simpler. Due to the fact that we can choose a basis where the 3-variables  $l_3$  and  $k_3$  are free of exterior momenta anyhow, we can directly do our transformation without posing any restriction on the exterior momenta, overcoming the need of an analytic continuation at the end altogether.

Nevertheless, we like to stress, as we are deforming to a still nondefinite signature, we are not allowed to set the small imaginary parts of the propagators to zero. They reflect the fact that the Green functions of field theory are boundary values of analytic functions (where the boundary is approached in the limit  $\eta \rightarrow 0$  [8, 9]). As we will make use of standard results of complex analysis in further steps, we have to keep the  $\eta$  prescription until the end of our calculation to be allowed to handle the integrand as an analytic function.

Having this in mind, let us investigate the behaviour of two-, three-, and four-point two-loop functions now. They differ only by the dimension of the parallel space [5, 4, 10] and the number of propagators contributing to  $N_l, N_k, N_{l,k}$ . With appropriate partial fractions done in  $N_l, N_k, N_{l,k}$  separately we can reduce all cases to sums of terms of the following structure:

$$\int d^4l \int d^4k \frac{1}{\Pi} \frac{1}{P_l P_{l,k} P_k},$$

with  $P_l \in N_l, P_{l,k} \in N_{l,k}, P_k \in N_k$ .

Here  $\Pi$  is a function of parallel space variables only and is a product of differences of propagators. We can use translation invariance to bring the propagators into a standard form:

$$\begin{aligned} P_l &= l_0^2 - l_1^2 - l_2^2 + l_3^2 - m_l^2 + i\eta \\ P_{l,k} &= (l_0 + k_0)^2 - (l_1 + k_1)^2 - (l_2 + k_2)^2 + (l_3 + k_3)^2 - m_{l,k}^2 + i\eta \\ P_k &= k_0^2 - k_1^2 - k_2^2 + k_3^2 - m_k^2 + i\eta. \end{aligned}$$

Next we use translation invariance once more to do the following shifts:

$$\begin{aligned} l_0 &\rightarrow l_0 + l_1, \quad l_3 \rightarrow l_3 + l_2, \\ k_0 &\rightarrow k_0 + k_1, \quad k_3 \rightarrow k_3 + k_2, \end{aligned}$$

which results in a representation of the form

$$\frac{\int d^4l \int d^4k \frac{1}{\tilde{\Pi}} \frac{1}{(l_0^2 + 2l_0l_1 + 2l_2l_3 + l_3^2 - m_l^2 + i\eta)}}{1} \frac{1}{((l_0 + k_0)^2 + 2(l_0 + k_0)(l_1 + k_1) + 2(l_2 + k_2)(l_3 + k_3) + (l_3 + k_3)^2 - m_{l,k}^2 + i\eta)} \frac{1}{(k_0^2 + 2k_0k_1 + 2k_2k_3 + k_3^2 - m_k^2 + i\eta)},$$

with some translated  $\tilde{\Pi}$ .

We recognize that the propagators are all linear in the middle variables  $l_1, l_2, k_1, k_2$ . The coefficients depend solely on the edge variables  $l_0, l_3, k_0, k_3$ . In particular the location of the imaginary parts of the propagators, considered as functions of the middle variables, depends on the sign of these coefficients. That is, for positive  $l_0, k_0, l_3, k_3$  resp.  $(l_0 + k_0), (l_3 + k_3)$  the middle variables  $l_1, k_1, l_2, k_2$  have their poles in the lower half-plane and vice versa.

We now intend to do the integration in the middle variables via the residue theorem. We choose to close in the upper half-plane for all four variables, which imposes constraints on the domain of integration for the edge variables [4, 2]. This is so because only for certain domains of the edge variables the poles of our propagators will be located in the upper half-plane, as we have just seen. We have to change the order of integration to do the integration in the four middle variables  $l_1, l_2, k_1, k_2$  first. We are allowed to do so because the integral over the modulus of our integrand exists

$$\int d^4l \int d^4k \frac{1}{|\tilde{\Pi}|} \frac{1}{|P_l|} \frac{1}{|P_{l,k}|} \frac{1}{|P_k|} < \infty,$$

as can be easily checked by explicit calculation. This is not necessarily so (for example doing the same steps as above) in the one-loop case  $N_k = N_{l,k} = 1$ ; one runs into trouble for the two- and three-point case. There, the low dimensional parallel space results in a degenerate  $\tilde{\Pi}$  which does not depend on  $l_2, l_3$  so that we cannot exchange the  $l_3, l_2$  integration. In the two-loop case this degeneracy is cancelled by the mixed propagator  $P_{l,k}$ . So in the case considered here we can interchange the orders of integration and obtain a result as a sum over terms of the form

$$\int_{d_{l_3}}^{u_{l_3}} \int_{d_{k_3}}^{u_{k_3}} \int_{d_{l_0}}^{u_{l_0}} \int_{d_{k_0}}^{u_{k_0}} dk_0 dl_0 dk_3 dl_3 \frac{1}{Q(l_0, k_0, l_3, k_3)}$$

where the upper and lower boundaries ( $u, \dots, d, \dots$ ) are determined by the constraints mentioned above and the new function  $Q$  is a quadratic form in the edge variables.

The actual form of  $Q$ , the boundaries and the number of contributing terms are governed by the topology and number of external particles coupling to the topology of Fig.(1) but the above structure is generic to all two-loop functions. It is only the fact that the function  $\Pi$  can be kept clear of all middle variables in the case of the two-point function which allows for such an easy integral representation (integrating out the edge variables  $l_3, k_3$ ) of this master function [5]. The factor

$$\frac{1}{(P_1 - P_2)(P_4 - P_5)}$$

appearing there is just the function  $\Pi$  depending on the two edge-variables  $l_0, k_0$  arising here.

Neither for the three- nor for the four-point function does such a simplification take place. Instead we have to integrate our quadratic forms  $Q$  which gives us integral representations of the form

$$\int \int \int \frac{\log(Q_1 + \sqrt{Q_2})}{\sqrt{Q_2} Q_3},$$

with quadratic forms  $Q_i$  for both cases. Integral representations of this kind are familiar for the three-point case [4]. The only difference between the three- and four-point functions is that in the case of the four-point function  $\tilde{\Pi}$  is not free of the middle variables  $l_2$  and  $k_2$ . But this only results in a proliferation of terms (more contributing residues) compared with the three-point functions, while the structure of the final integral representation remains unchanged.

It was the objective of this short note to show that similar integral representations appear for the case of three- and four-point functions. In the three-point case, these integral representations are the starting point for numerical approaches valid in arbitrary kinematical regimes [2], and we hope that we can obtain similar results for the box-functions in the future [3], following the recipe as outlined here.

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